COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Nonstationary optimal paths and tails of prehistory probability density in multistable stochastic systems"

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A possible criterion to select the optimal path when more than a nonstationary path is found in multistable stochastic systems is discussed. Under this criterion, some of the nonstationary paths derived in Phys. Rev. E **55**, 5338 (1997) would not be optimal. [S1063-651X(99)04602-4]

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In a recent paper [1], the problem of nonstationary optimal paths in multistable stochastic systems is addressed. The authors find that if the problem of the optimal path is studied over a finite time period, or assuming that the starting or final point differs from the stationary ones, the path which connects the initial to the final point can be markedly different from the stationary optimal one.

The solution of the problem proposed in [1] consists in solving a Hamiltonian with appropriate boundary conditions, given over a finite time span (as opposed to an infinite time range as in the stationary case). These boundary conditions are, however, given in part at the initial time and in part at the final time: in general, this implies that the solution of the Hamiltonian equations may not be unique. For the sake of argument, and to make contact with [1], I consider the one-dimensional system

$$\dot{x} = -U'(x) + \xi(t), \quad \langle \xi(t)\xi(s)\rangle = 2D\delta(t-s), \tag{1}$$

where $U(x) = -x^2/2 + x^4/4$ and $\xi(t)$ is a Gaussian process with average zero. The associated Hamiltonian yields the equations of motion and boundary conditions

$$\ddot{x} = U'(x)U''(x),$$
$$x(t_0) = x_0, \quad x(t_f) = x_f. \tag{2}$$

Given that the solution of Eq. (2) is not unique, a criterion must be given to pick the "correct" optimal path. In the case of a stationary process, one selects the solution of minimal action, among all possible solutions. It may be argued that, in the spirit of [1], the same approach ought to be followed for nonstationary problems. I define the action S as $S \equiv \int [\dot{x} + U'(x)]^2/2 dt$.

If this is the case, however, some of the nonstationary paths derived in [1] are not the optimal ones: in particular, there may exist a path which has a lower action than the path no. 3 in Fig. 2 of [1]: a path like the no. 3 path in [1] drops to the stable state (with negligible contribution to the action) and then rises to the final point, with a contribution to the action which is similar to the contribution coming from a stationary optimal path between the stable state and the final point. Another possible path connecting the same initial and

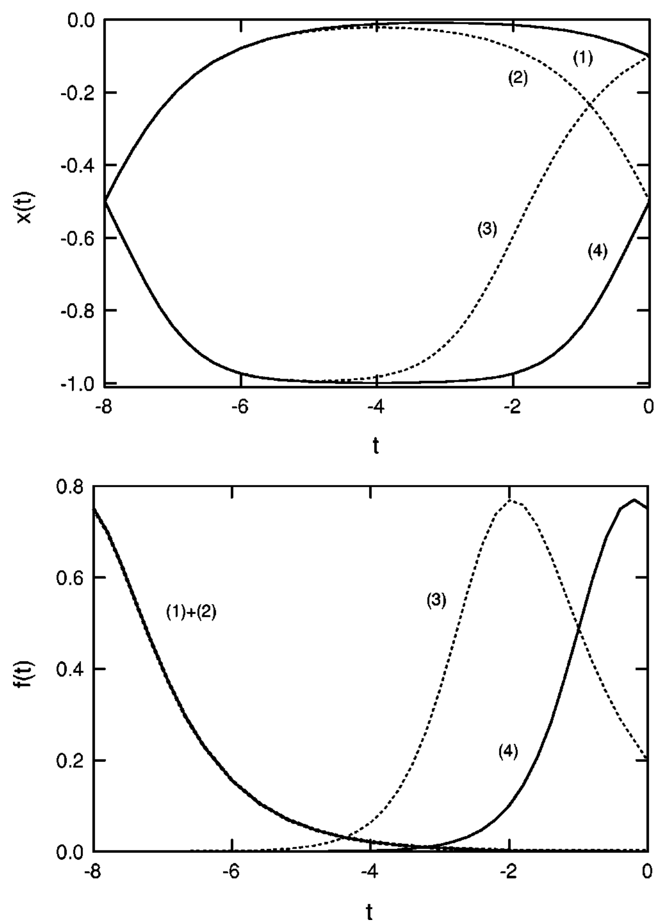

FIG. 1. Top: paths solving Eq. (2), for two different final points and the same initial point. Bottom: corresponding optimal fields. The optimal (nonoptimal) curves are shown as solid (dashed) lines. The numbers identify path and corresponding optimal field.

final points is the path which moves directly from the initial point climbing to the unstable state (with some sizable contribution to the action) stays on the unstable state for some time and then drops from the unstable state to the final point: these last two stages give a negligible contribution to the action. Clearly, if the action of the latter path were smaller than the action of the former path, then the latter path would be the optimal nonstationary path, assuming that the least action criterion is applicable.

The existence of couples of nonstationary paths is confirmed by direct numerical solution of Eq. (2). I report in Fig. 1 (top) some nonstationary paths corresponding to the same starting point of path no. 3 in [1], over the same time range, for two different final points: with a solid line I drew the optimal nonstationary paths and with a dashed line the other possible (nonoptimal) path connecting the initial and final points. Still in Fig. 1 (bottom), I drew the optimal field $[f(t)]$ for each path (see [1] for details). The action can also be evaluated as the integral of $f(t)^2$ over the time domain: it is clear by inspection that the field corresponding to (1) yields a lower action than the field corresponding to (3). This is confirmed by the numerics: for (1), $S=0.220$, whereas for (3), $S=0.490$.

As mentioned, whether the nonstationary optimal path will move towards the stable or unstable state, before reaching the final point, depends on the initial and final points, beside the time range. The situation in which the path first moves to the stable state and then towards the final point is shown in Fig. 1 with the curves labeled (2) and (4): the solid lines are the optimal ones.

It is also possible to map out the region for which one scenario is preferred to the others, which I summarize in Fig. 2 (computed for a time range equal to 8). For initial (x_0) and final (x_f) points which lie to the left of and below the line, the optimal path will first move towards the stable state before it climbs to the final point. Note that the solid line in Fig. 2 is indistinguishable from the line one would obtain imposing the condition $U(0)-U(x_0)=U(x_f)-U(-1)$.

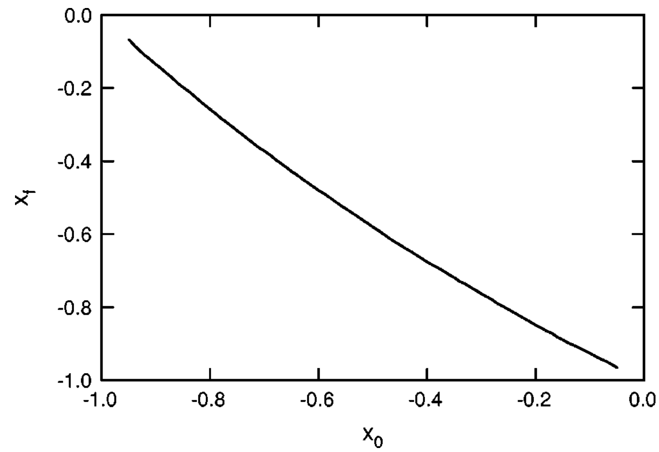

FIG. 2. Switching line between a path like no. 1 and no. 4 in Fig. 1, as function of initial (x_0) and final (x_f) points, for the time range used in Fig. 1.

This means, physically, that for the time range considered, the path which is optimal would be dictated by the static potential differences between initial and final states and stationary states.

In conclusion, the problem put forward in [1] is very interesting; a somewhat open question is the criterion which should be followed to pick the optimal nonstationary path in case more than one path satisfies the boundary conditions: if the same criterion (minimal action) used for the stationary case is invoked, then some of the paths computed in [1] are not the optimal ones. With the criterion put forward in this Comment, the switching line between different behaviors is worked out. It is an open question whether some of the optimal paths proposed here and the switching line should be visible in real systems, and also the relation between the present switching line and the switching line found in nonequilibrium systems [2].

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